

On recovering missing values in a pathwise setting

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Abstract

The paper suggests a frequency criterion of error-free recoverability of a missing value for sequences, i.e., discrete time processes, in a pathwise setting, without using probabilistic assumptions on the ensemble. This setting targets situations where we deal with a sole sequence that is deemed to be unique and such that we cannot rely on statistics collected from other similar samples. A missing value has to be recovered using the intrinsic properties of this sole sequence. In the paper, error-free recoverability is established for classes of sequences with Z-transform vanishing at a point with a rate that can be chosen arbitrarily. The corresponding recovering kernels are obtained explicitly. Some robustness with respect to noise contamination is established for the suggested recovering algorithm.

Key words: data recovery, minimal Gaussian processes, pathwise criteria, frequency criterion, discrete time, Z-transform, robustness

1 Introduction

A core problem of the mathematical theory of signal processing is the problem of recovery of missing data. For continuous data, the recoverability is associated with smoothness or analytical properties of the processes. For discrete time processes, it is less obvious how to interpret analyticity; so far, these problems were studied in a stochastic setting, where an observed process is deemed to be representative of an ensemble of paths with the probability distribution that is either known or can be estimated from repeating experiments. A classical result for stationary stochastic processes with the spectral density ϕ is that a missing single value is recoverable with zero error if and only if

$$\int_{-\pi}^{\pi} \phi(e^{i\omega})^{-1} d\omega = -\infty. \quad (1)$$

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(Kolmogorov [9], Theorem 24). Stochastic stationary Gaussian processes without this property are called *minimal* [9]. In particular, a process is recoverable if it is “band-limited” meaning that the spectral density is vanishing on an arc of the unit circle $\mathbb{T} = \{z \in \mathbf{C} : |z| = 1\}$. This illustrates the relationship of recoverability on the notion of bandlimitness or its relaxed versions such as (1). In particular, criterion (1) was extended on stable processes [10] and vector Gaussian processes [11]. There are many works devoted to minimization of data recovery errors in different settings; see e.g. [1, 2, 4, 5, 3, 12, 13].

The paper suggests a criterion of error-free recoverability of sequences (discrete time processes) similar to (1) based on intrinsic properties of sequences, in the pathwise setting, without using probabilistic assumptions on the ensemble. This setting targets situations where we deal with a sole sequence that is deemed to be unique and such that one cannot rely on statistics collected from observations of other similar samples. An estimate of the missing value has to be done based on the intrinsic properties of this sole sequence and the observed values. The result is based on the approach developed for pathwise predicting [6, 7, 8], where some predictors were derived to establish error-free predicability. In the present paper, error-free recoverability is established for certain classes of processes with Z-transform vanishing at a point (Theorem 1). The decay rate can be selected as an arbitrarily small power of the distance to this point (Proposition 1). The corresponding recovering kernels are obtained and represented explicitly via their transfer functions. Some robustness with respect to noise contamination is established for the suggested recovering algorithm.

2 Some definitions and background

Let $\mathbb{T} \triangleq \{z \in \mathbf{C} : |z| = 1\}$, and let \mathbb{Z} be the set of all integers.

We denote by ℓ_r the set of all sequences $x = \{x(t)\} \subset \mathbf{C}$, $t = 0, \pm 1, \pm 2, \dots$, such that $\|x\|_{\ell_r} = (\sum_{t=-\infty}^{\infty} |x(t)|^r)^{1/r} < +\infty$ for $r \in [1, \infty)$ or $\|x\|_{\ell_\infty} = \sup_t |x(t)| < +\infty$ for $r = +\infty$.

For $x \in \ell_1$ or $x \in \ell_2$, we denote by $X = \mathcal{Z}x$ the Z-transform

$$X(z) = \sum_{t=-\infty}^{\infty} x(t)z^{-t}, \quad z \in \mathbf{C}.$$

Respectively, the inverse $x = \mathcal{Z}^{-1}X$ is defined as

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega t} d\omega, \quad t = 0, \pm 1, \pm 2, \dots$$

We have that $x \in \ell_2$ if and only if $\|X(e^{i\omega})\|_{L_2(-\pi, \pi)} < +\infty$. In addition, $\|x\|_{\ell_\infty} \leq$

$$\|X(e^{i\omega})\|_{L_1(-\pi, \pi)}.$$

We use the sign \circ for convolution in ℓ_2 .

Let $\widehat{\mathcal{K}}$ be the set of all real sequences $\widehat{k} \in \ell_2$ such that $\widehat{k}(0) = 0$.

By recoverability, we mean a possibility of constructing a linear recovering operator as described in the definition below.

Definition 1 *Let $\mathcal{Y} \subset \ell_2$ be a class sequences.*

(i) *We say that this class is recoverable if there exists a sequence $\{\widehat{k}_m(\cdot)\}_{m=1}^{+\infty} \subset \widehat{\mathcal{K}}$ and*

$$|x(0) - \widehat{x}_m(0)| \rightarrow 0 \quad \text{as } m \rightarrow +\infty \quad \forall x \in \mathcal{Y}.$$

$$\text{Here } \widehat{x} = h \circ x, \text{ i.e. } \widehat{x}_m(0) \triangleq \sum_{s \in \mathbb{Z} \setminus \{0\}} \widehat{k}_m(s)x(s).$$

(ii) *We say that the class \mathcal{Y} is uniformly recoverable if, for any $\varepsilon > 0$, there exists $\widehat{k}(\cdot) \in \widehat{\mathcal{K}}$ such that*

$$|x(0) - \widehat{x}(0)| \leq \varepsilon \quad \forall x \in \mathcal{Y}.$$

$$\text{Here } \widehat{x} = h \circ x, \text{ i.e. } \widehat{x}(0) \triangleq \sum_{s \in \mathbb{Z} \setminus \{0\}} \widehat{k}(s)x(s).$$

3 The main results

We will establish recoverability for sequences with Z-transforms in a special weighted L_p -spaces.

Let us introduce some classes of the corresponding weights.

Special weighted spaces

For $p \in (1, +\infty]$, let $\mathcal{H}(p)$ be the set of functions $h : (-\pi, \pi) \rightarrow (0, +\infty)$ such that the following holds:

(h1) h is continuous at any $\omega \in (-\pi, \pi)$, $\inf_{\omega \in (-\pi, \pi)} h(\omega) > 0$, and $h(-\omega) = h(\omega)$ for all ω .

(h2) There exists a function $W : (-\pi, \pi) \rightarrow \mathbf{R}$ such that $W(\omega) = W(-\omega)$, and that

$$\begin{aligned} \int_0^\pi |W(\omega)|^q h(\omega)^{-q} d\omega < +\infty \quad \text{for } q = (1 - 1/p)^{-1}, \\ \int_{\pi-\varepsilon}^\pi W(\omega) d\omega = +\infty \quad \forall \varepsilon \in (\pi - 1, \pi). \end{aligned}$$

The following proposition gives examples of possible choices for $h \in \mathcal{H}(p)$ and of the corresponding functions W .

Proposition 1 (i) *Let a function $h : (-\pi, \pi) \rightarrow (0, +\infty)$ be such that condition (h1) outlined in the definition of $\mathcal{H}(p)$ holds, and that there exists $a \in \mathbf{R}$ and $q \geq 1$ such that*

$$\int_{-\pi}^{\pi} h(\omega)^a d\omega = +\infty, \quad \int_{-\pi}^{\pi} h(\omega)^{q(a-1)} d\omega < +\infty.$$

Then $h \in \mathcal{H}(p)$ for $p = (1 - 1/q)^{-1}$. A possible choice of W is $W(\omega) = h(\omega)^a$.

(ii) *Let $\nu > 0$, and let $h(\omega) = (\pi^2 - \omega^2)^{-\nu}$ for $\omega \in (-\pi, \pi)$. Then $h \in \mathcal{H}(p)$ for any $p > 1/\nu$. A possible choice of W is $W(\omega) = h(\omega)^{1/\nu}$. In particular, for any $\nu > 0$, any function h of this type belongs to $\mathcal{H}(\infty)$.*

(iii) *Let a function $h : (-\pi, \pi) \rightarrow (0, +\infty)$ be such that conditions (h1) outlined in the definition of $\mathcal{H}(p)$ holds, and that $\int_{-\pi}^{\pi} h(\omega) d\omega = +\infty$. Then $h \in \mathcal{H}(\infty)$. A possible choice of W is $W(\omega) = h(\omega)$.*

For $p \in (1, +\infty]$ and $h \in \mathcal{H}(p)$, let $\mathcal{X}(p, h)$ be the class of all sequences $x \in \ell_2$ such that

$$\int_{-\pi}^{\pi} h(\omega) |X(e^{i\omega})|^p d\omega < +\infty, \quad \text{if } p \in [1, +\infty), \quad (2)$$

and

$$\text{ess sup}_{\omega \in (-\pi, \pi)} h(\omega) |X(e^{i\omega})| < +\infty, \quad \text{if } p = +\infty, \quad (3)$$

where $X = \mathcal{Z}x$.

Note that if $h \in \mathcal{H}(p)$ then $h(\omega) \rightarrow +\infty$ as $\omega \rightarrow \pm\pi$. In this case, (2) and (3) hold for “degenerate” processes, with $X(e^{i\omega})$ vanishing as $\omega \rightarrow \pm\pi$ with the corresponding rate of decay defined by h . In particular, a class $\mathcal{X}(\nu, p)$ includes all band-limited processes x such that $X(e^{i\omega}) \in L_p(-\pi, \pi)$ and $X(e^{i\omega}) = 0$ for $\omega \notin [-\Omega, \Omega]$, for some $\Omega \in (0, \pi)$, where $X = \mathcal{Z}x$.

It can be also noted that if $h_1 \in \mathcal{H}(p)$ and $h_2 \in \mathcal{H}(p)$ be such that $h_1(\omega) \leq Ch_2(\omega)$ for all ω and for some $C > 0$, then $\mathcal{X}(p, h_2) \subset \mathcal{X}(p, h_1)$.

For given $p \in [1, +\infty]$ and $h \in \mathcal{H}(p)$, let $\mathcal{U}(p, h)$ be a class of processes $x \in \mathcal{X}(p, h)$ such that

$$\int^{D_\varepsilon} h(\omega) |X(e^{i\omega})|^p d\omega \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0+ \quad \text{uniformly over } x \in \mathcal{U}(p, h),$$

for $p \in [1, +\infty)$, and

$$\operatorname{ess\,sup}_{\omega \in D_\varepsilon} h(\omega) |X(e^{i\omega})|^p d\omega \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 + \quad \text{uniformly over } x \in U(p, h).$$

for $p = +\infty$. Here $D_\varepsilon = (-\pi, -\pi + \varepsilon) \cup (\pi - \varepsilon, \pi)$ and $X = \mathcal{Z}x$.

Recovering kernels and recoverable sequences

Lemma 1 *Let $p \in (1, +\infty]$ and $h \in \mathcal{H}(p)$ be given, and let a function $W : (-\pi, \pi) \rightarrow \mathbf{R}$ be such as described in the definition for $\mathcal{H}(p)$, with this h . For $n > 1$, consider kernels constructed as $\widehat{k}(\cdot) = \widehat{k}_n(\cdot) = \mathcal{Z}^{-1} \widehat{K}_n$, where functions $\widehat{K} = \widehat{K}_n$ are such that*

$$\begin{aligned} \widehat{K}(e^{i\omega}) &= 1, & |\omega| < \pi - 1/n, \\ \widehat{K}(e^{i\omega}) &= -W(\omega), & |\omega| \in [\pi - 1/n, \pi - \varepsilon_n], \\ \widehat{K}(e^{i\omega}) &= 0, & |\omega| \in (\pi - \varepsilon_n, \pi). \end{aligned}$$

Here $\varepsilon_n \in (0, 1/n)$ are uniquely defined such that

$$\int_{\pi-1/n}^{\pi-\varepsilon_n} W(\omega) d\omega = \pi - \frac{1}{n}. \quad (4)$$

Then $\widehat{k} \in \widehat{\mathcal{K}}$, i.e. $\widehat{k} \in \ell_2$ and $\widehat{k}(0) = 0$.

It follows that the kernels introduced in Lemma 1 are potential candidates for the role of recovering kernels presented in Definition 1. The following theorem shows that these kernels ensure recoverability of certain classes of sequences.

Theorem 1 *For any $p \in (1, +\infty)$ or $p = +\infty$, and for any $h \in \mathcal{H}(p)$, the following holds.*

- (i) *The class $\mathcal{X}(p, h)$ is recoverable.*
- (ii) *The class $\mathcal{U}(p, h)$ is uniformly recoverable.*
- (iii) *The kernels $\widehat{k} = \widehat{k}_n$ introduced in Lemma 1 ensure prediction required in (i) and (ii) as $n \rightarrow +\infty$. For these kernels,*

$$|x(0) - \widehat{x}(0)| \rightarrow 0 \quad \text{as } n \rightarrow +\infty \quad \forall x \in \mathcal{X}(p, h).$$

Moreover, for any $\varepsilon > 0$, there exists $n > 0$ such that

$$|x(0) - \hat{x}(0)| \leq \varepsilon \quad \forall x \in \mathcal{U}(p, h). \quad (5)$$

Here $\hat{x}(t) \triangleq \sum_{s=-\infty}^{\infty} \hat{k}(s)x(s)$.

Remark 1 The choice of the functions W and recovering kernels \hat{k} is not unique for a given $h \in \mathcal{H}(p)$. For instance, let h be such as described in Proposition 1(ii), then $W(\omega) = h(\omega, \nu)^\mu$ satisfies the conditions outlined in the definition for $\mathcal{H}(p)$ for any $\mu > 0$ such that $\nu\mu \in \left[1, 1 + \frac{1}{q}\right)$, where $q = (1 - 1/p)^{-1}$.

4 Proofs

Proof of Proposition 1. To prove statement (i), it suffices to observe that $W(\omega) = h(\omega)^a$ satisfy condition (h2) outlined in the definition of $\mathcal{H}(p)$. For statements (ii) and (iii), it suffices to observe that $h(\omega)$ satisfy the assumptions of statement (i). For statement (ii), we have to select $a = 1/\nu$. For statement (ii), we have to select $a = 1$. \square

Proof of Lemma 1. Since $\widehat{K}(e^{-i\omega}) = \overline{\widehat{K}(e^{i\omega})}$, we have that $\hat{k} = \mathcal{Z}^{-1}\widehat{K}$ is real valued. The conditions (i)-(ii) in the definition of $\mathcal{H}(p)$ ensure that there exists $\varepsilon_n \in (0, 1/n)$ such that (4) holds. Further, (4) implies that

$$\int_{-\pi+\varepsilon_n}^{-\pi+1/n} \widehat{K}(\omega) d\omega + \int_{\pi-1/n}^{\pi-\varepsilon_n} \widehat{K}(\omega) d\omega = 2/n - 2\pi = - \int_{-\pi+1/n}^{\pi-1/n} \widehat{K}(e^{i\omega}) d\omega.$$

Hence $\int_{-\pi}^{\pi} \widehat{K}(e^{i\omega}) d\omega = 0$. Therefore,

$$\hat{k}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{K}(e^{i\omega}) d\omega = 0.$$

This completes the proof of Lemma 1. \square

Proof of Theorem 1. Let $n \rightarrow +\infty$, and let $\widehat{K} = \widehat{K}_n$ be as defined above. Let $x \in \mathcal{X}(p, h)$, $X \triangleq \mathcal{Z}x$, $\hat{k} = \mathcal{Z}^{-1}\widehat{K}$, and

$$\hat{x}(t) \triangleq \sum_{s=-\infty}^t \hat{k}(t-s)x(s).$$

By the definitions, it follows that $\widehat{X}(e^{i\omega}) \triangleq \widehat{K}(e^{i\omega}) X(e^{i\omega}) = (\mathcal{Z}\hat{x})(e^{i\omega})$.

We have that $\|\widehat{X}(e^{i\omega}) - X(e^{i\omega})\|_{L_1(-\pi, \pi)} = I_1 + I_2 + I_3$, where

$$I_k = \int_{D_k} |\widehat{X}(e^{i\omega}) - X(e^{i\omega})| d\omega = \int_{D_k} |(\widehat{K}(e^{i\omega}) - 1)X(e^{i\omega})| d\omega,$$

and where

$$\begin{aligned} D_1 &= (-\pi + 1/n, \pi - 1/n), \quad D_2 = [-\pi + \varepsilon_n, -\pi + 1/n] \cup [\pi - 1/n, \pi - \varepsilon_n], \\ D_3 &= (-\pi, -\pi + \varepsilon_n) \cup (\pi - \varepsilon_n, \pi). \end{aligned}$$

Clearly,

$$I_1 = \|(\widehat{K}(e^{i\omega}) - 1)X(e^{i\omega})\|_{L_1(D_1)} = 0.$$

Let us estimate I_2 . By the assumptions, $\|h(\omega)X(e^{i\omega})\|_{L_p(-\pi, \pi)} < +\infty$. We have that

$$I_2 = \int_{D_2} |(\widehat{K}(e^{i\omega}) - 1)X(e^{i\omega})| d\omega \leq \psi_n \|X(e^{i\omega})\|_{L_p(-\pi, \pi)} \leq \psi_n \|h(\omega)X(e^{i\omega})\|_{L_p(D_2)},$$

where

$$\psi_n = \left(\int_{D_2} |h(\omega)|^{-q} |\widehat{K}(e^{i\omega}) - 1|^q d\omega \right)^{1/q},$$

and where $q = (1 - 1/p)^{-1} \in [1, +\infty)$. We have that

$$\psi_n \leq \left[\int_{D_2} |h(\omega)|^{-q} |\widehat{K}(e^{i\omega})|^q d\omega \right]^{1/q} + \left[\int_{D_2} |h(\omega)|^{-q} d\omega \right]^{1/q}.$$

Clearly,

$$\int_{D_2} |h(\omega)|^{-q} d\omega \leq \int_{-\pi}^{\pi} |h(\omega)|^{-q} d\omega < +\infty.$$

By the assumption on \widehat{K} ,

$$\int_{D_2} |h(\omega)|^{-q} |\widehat{K}(e^{i\omega})|^q d\omega \leq 2 \int_0^{\pi} |h(\omega)|^{-q} |W(\omega)|^q d\omega.$$

By the assumptions on W , it follows that the sequence $\{\psi_n\}$ is bounded. Hence $I_2 \rightarrow 0$ as $n \rightarrow +\infty$.

Let us estimate I_3 . We have that

$$I_3 = \int_{D_3} |(\widehat{K}(e^{i\omega}) - 1)X(e^{i\omega})| d\omega = \int_{D_3} |X(e^{i\omega})| d\omega \rightarrow 0 \quad n \rightarrow +\infty.$$

It follows that $I_1 + I_2 + I_3 \rightarrow 0$ for any $c > 0$ and $x \in \mathcal{X}(h, p)$. Hence $\|\widehat{y} - y\|_{\ell_\infty} \rightarrow 0$ as $n \rightarrow +\infty$ for any $x \in \mathcal{X}(h, p)$. This completes the proof of statement (i).

Let us prove statement (ii). By the assumptions on $\mathcal{U}(h, p)$, it follows that, in the notations of the proof above,

$$\begin{aligned} \|\widehat{X}(e^{i\omega}) - X(e^{i\omega})\|_{L_1(-\pi, \pi)} &= I_1 + I_2 + I_3 \\ &= I_2 + I_3 \rightarrow 0 \quad \text{as } n \rightarrow +\infty \end{aligned}$$

uniformly over $x \in \mathcal{U}(h, p)$. For any $\varepsilon > 0$, one can select n such that $I_2 + I_3 \leq \varepsilon$. This choice ensures that $\|\widehat{x} - x\|_{\ell_\infty} \leq \varepsilon$. This completes the proof of statement (ii). It follows that the predicting kernels $\widehat{k}(\cdot) = \mathcal{Z}^{-1}\widehat{K}$ are such as required. This completes the proof of Theorem 1. \square

5 On robustness with respect to noise contamination

Let us discuss the impact of on the recovery error of the presence of the noise contaminating recoverable sequences. Assume that the kernels \widehat{k} described above and designed for recoverable sequences are applied to a sequence with a noise contamination. Let us consider an input sequence $x \in \ell_2$ such that $x = x_0 + \eta$, where $x_0 \in \mathcal{U}(p, h)$ for some $h \in \mathcal{H}(p)$ and $p \in [1, +\infty]$, and where $\eta \in \ell_2 \setminus \mathcal{X}(p, h)$ represents a noise. Let $X = \mathcal{Z}x$, $X_0 = \mathcal{Z}x_0$, and $N = \mathcal{Z}\eta$. We assume that $X_0(e^{i\omega}) \in L_1(-\pi, \pi)$ and $\|N(e^{i\omega})\|_{L_1(-\pi, \pi)} = \sigma$. The parameter $\sigma \geq 0$ represents the intensity of the noise.

Assume that the kernels \widehat{k} are such as described in Theorem 1. In the proof of Theorem 1, we found that, for an arbitrarily small $\varepsilon > 0$, there exists $n = n(p, h)$ such that

$$\int_{-\pi}^{\pi} |(\widehat{K}(e^{i\omega}) - 1)X_0(e^{i\omega})| d\omega \leq 2\pi\varepsilon \quad \forall x_0 \in \mathcal{U}(p, h),$$

where $\widehat{K} = \mathcal{Z}\widehat{k}$. For $\widehat{x}_0 = \widehat{k} \circ x_0$, this implies that

$$|\widehat{x}_0(0) - x_0(0)| \leq \|\widehat{x}_0 - x_0\|_{\ell_\infty} \leq \varepsilon.$$

Let us estimate the prediction error for the case where $\sigma > 0$. For $\widehat{x} = \widehat{k} \circ x$, we have We

have that

$$\|\hat{x} - x\|_{\ell_\infty} \leq E_0 + E_\eta,$$

where

$$E_0 = \frac{1}{2\pi} \|(\hat{K}(e^{i\omega}) - 1)X_0(e^{i\omega})\|_{L_1(-\pi, \pi)} \leq \varepsilon, \quad E_\eta = \frac{1}{2\pi} \|(\hat{K}(e^{i\omega}) - 1)N(e^{i\omega})\|_{L_1(-\pi, \pi)}.$$

The value E_η represents the additional error caused by the presence of unexpected high-frequency noise (when $\sigma > 0$). It follows that

$$|\hat{x}(0) - x(0)| \leq \|\hat{x} - x\|_{\ell_\infty} \leq \varepsilon + \sigma(\kappa + 1), \quad (6)$$

where $\kappa = \sup_{\omega \in [-\pi, \pi]} |\hat{K}(e^{i\omega})|$.

Therefore, it can be concluded that the prediction is robust with respect to noise contamination for any given ε .

It can be noted that if $\varepsilon \rightarrow 0$ then $n \rightarrow +\infty$ and $\kappa \rightarrow +\infty$. In this case, error (6) is increasing for any given $\sigma > 0$. This happens when the predictor is targeting too small a size of the error for the sequences from $\mathcal{X}(p, h)$, i.e., under the assumption that $\sigma = 0$.

The equations describing the dependence of ε and κ on n could be derived similarly to estimates in [7], Section 6, where it was done for some predicting kernels and for band-limited sequences.

6 Discussion and future development

The paper suggests a frequency criterion of error-free recoverability of a single missing value in pathwise deterministic setting in the spirit of the Kolmogorov's criterion of minimality for stochastic Gaussian stationary processes [9]. Some robust recovering algorithm for classes of these sequences is suggested.

With the choice of h from Proposition 1 (ii), Theorem 1 gives a criterion that reminds the classical Kolmogorov's criterion (1) for the spectral densities [9]. However, the degree of similarity is quite limited. For instance, if a stationary Gaussian process has the spectral density $\phi(\omega) \geq \text{const} \cdot (\pi^2 - \omega^2)^\nu$ for $\nu \in (0, 1)$, then, according to criterion (1), this process is not minimal [9], i.e. this process is non-recoverable. On the other hand, Theorem 1 and Proposition 1 (ii) imply that, for $p \in (1, +\infty]$, the class $\mathcal{X}(p, h)$ is recoverable with $|h(\omega)| = (\pi^2 - \omega^2)^{-\nu}$ if $\nu > 1/p$; in particular, this class includes sequences x such that $|X(e^{i\omega})| \leq \text{const} \cdot (\pi^2 - \omega^2)^\nu$ for

$X = \mathbb{Z}x$. If $p = +\infty$ then recoverability can be ensured with an arbitrarily small rate of decay $\nu > 0$. Nevertheless, this similarity still could be used for analysis of the properties of pathwise Z-transforms for stochastic Gaussian processes. In particular, assume that $y = \{y(t)\}_{t \in \mathbb{Z}}$ is a stochastic stationary Gaussian process with spectral density ϕ such that (1) does not hold. It follows that adjusted paths $\{(1 + \delta t^2)^{-1}y(t)\}_{t \in \mathbb{Z}}$, where $\delta > 0$, cannot have Z-transform satisfying the assumption of Proposition 1 (ii). We leave this analysis for the future research.

There are other open questions. In particular, it is unclear if it is possible to obtain pathwise necessary conditions of error-free recoverability based on Z-transform. In addition, it could be interesting to find other convenient choices of functions h and \hat{K} , similarly to Proposition 1 and Lemma 1.

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References

- [1] Y. Alem, Z. Khalid, R.A. Kennedy. (2014). Band-limited extrapolation on the sphere for signal reconstruction in the presence of noise, *Proc. IEEE Int. Conf. ICASSP'2014*, pp. 4141-4145.
- [2] T. Cai, G. Xu, and J. Zhang. (2009), On recovery of sparse signals via ℓ_1 minimization, *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3388-3397.
- [3] D. L. Donoho and P. B. Stark. (1989). Uncertainty principles and signal recovery. *SIAM J. Appl. Math.*, vol. 49, no. 3, pp. 906–931.
- [4] E. Candés, T. Tao. (2006), Near optimal signal recovery from random projections: Universal encoding strategies? *IEEE Transactions on Information Theory* 52(12) (2006), 5406-5425.
- [5] E.J. Candes, J. Romberg, T. Tao. (2006). Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Transactions on Information Theory* **52** (2), 489–509.
- [6] N.Dokuchaev. (2012). On predictors for band-limited and high-frequency time series. *Signal Processing* **92**, iss. 10, 2571-2575.

- [7] N. Dokuchaev. (2012). Predictors for discrete time processes with energy decay on higher frequencies. *IEEE Transactions on Signal Processing* **60**, No. 11, 6027-6030.
- [8] N. Dokuchaev. (2016). Near-ideal causal smoothing filters for the real sequences. *Signal Processing* **118**, iss. 1, pp. 285-293.
- [9] A.N. Kolmogorov. (1941). Interpolation and extrapolation of stationary stochastic series. *Izv. Akad. Nauk SSSR Ser. Mat.*, 5:1, 3–14.
- [10] V. V. Peller (2000). Regularity conditions for vectorial stationary processes. *In: Complex Analysis, Operators, and Related Topics. The S.A. Vinogradov Memorial Volume*. Ed. V. P. Khavin and N. K. Nikol'skii. Birkhauser Verlag, pp 287-301.
- [11] M. Pourahmadi (1984). On minimality and interpolation of harmonizable stable processes. *SIAM Journal on Applied Mathematics*, Vol. 44, No. 5, pp. 1023–1030.
- [12] M. Pourahmadi. Estimation and interpolation of missing values of a stationary time series. *J. Time Ser. Anal.*, 10 (1989), pp. 149-169
- [13] J. Tropp and A. Gilbert. (2007). Signal recovery from partial information via orthogonal matching pursuit, *IEEE Trans. Inf. Theory*, vol. 53, no. 12, pp. 4655–4666.